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When F is a proper face of a (convex) polytope P in a Euclidean space E, an F-figure of P is an intersection of sets P and H $P \cap H$ where H is a hyperplane strictly separating F from all vertices of $(P \sim F)$ Here it is shown how, when the origin is interior to P, the combinatorial (and, in a sense, the projective) structure of P's F-figures can be described in terms of the boundary structure of the polar polytope P^0. The main tool is the notion of the adjoint of a projective transformation and a basic formula relating adjoints to polars.

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" sets P and F (which have a
one to one relation)

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ADJOINTS OF PROJECTIVE TRANSFORMATIONS

AND FACE-FIGURES OF CONVEX POLYTOPES

by

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Technical Report No. 59

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ADJOINTS OF PROJECTIVE TRANSFORMATIONS AND
FACE-FIGURES OF CONVEX POLYTOPES

VICTOR KLEE

Introduction. When F is a proper face of a (convex) polytope P in a Euclidean space E, an F-_figure_ of P is an intersection $P \cap H$ where H is a hyperplane strictly separating F from all vertices of $P \sim F$. Here it is shown how, when the origin is interior to P, the combinatorial (and, in a sense, the projective) structure of P's F-figures can be described in terms of the boundary structure of the polar polytope P^o. The main tool is the notion of the adjoint of a projective transformation and a basic formula relating adjoints to polars.

The adjoint of a projective transformation. Suppose that U and V are vector spaces paired by a bilinear form in such a way that each space distinguishes points of the other [3, pp. 137-138], and suppose the same is true of X and Y. All spaces are over the same ordered field and both bilinear forms are denoted by $\langle \, , \, \rangle$. The treatment in this section is purely algebraic, with no mention of topology, because we are concerned primarily with cases in which there is only one reasonable topology compatible with the algebraic

structure---namely, those in which $U = V$, $X = Y$, all spaces are Euclidean, and the bilinear forms are the usual inner products.

For any set W contained in one of two paired spaces, the polar set W^O is contained in the other; for example,

$$W \subset U \implies W^O = \{v \in V: w \in W \implies \langle w, v \rangle \leq 1\} \subset V.$$

Recall that two linear transformations $S: U \to X$ and $T: Y \to V$ are mutually adjoint if $\langle S(u), y \rangle = \langle u, T(y) \rangle$ for all $u \in U$ and $y \in Y$. In the study of convex sets, the usefulness of adjoints is due largely to a relationship with polarity that may be regarded as a sort of commutativity of the following diagram:
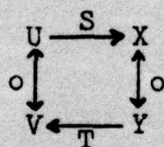
Specifically (cf. [2, p.51]),

$$W \subset U \implies T((SW)^O \cap (dmn\ T)) = W^O \cap (rng\ T).$$

Theorems 1 and 2 below extend this commutativity result to projective transformations and their adjoints. The treatment of projective transformations, being similar to that of [2][5], is rather unnatural from the viewpoint of projective geometry but is well suited to the study of convex sets. As is indicated later, particular instances of adjoints appear in [2][5], though they are not called that in the nonlinear case. Indeed, the results of [2][5] would suffice for the study of face-figures. However, it seems worthwhile to formulate the notion of adjoint and establish

the commutativity result in full generality, for they are useful in other situations as well.

Relative to the quadruple (U,V,X,Y) and the associated bilinear forms, a <u>projective transformation</u> $S\colon U \to X$ is defined as a transformation of the form

(1) $$S(\cdot) = \frac{S_o(\cdot) + S_1}{1 - \langle \cdot, S_2 \rangle}$$

where $S_o\colon U \to X$ is a linear transformation, S_1 is a point of X, and S_2 is a point of V (cf. [2, p.4][5, p.18]). We write $S\colon U \to X$, using three dots instead of two, to emphasize that the domain of S may be a proper subset of U. Of course the domain should be disjoint from the set $\{u \in U\colon \langle u, S_2 \rangle = 1\}$, but subject to that restriction various conventions are possible and the best choice may depend on the intended application. In studying the interplay of convexity, polarity and projectivity, the domain should be convex and include the origin 0, and for our present purpose it is convenient to define the domain of S as the largest set satisfying the stated restrictions. Thus

$$\operatorname{dmn} S = \{u \in U\colon \langle u, S_2 \rangle < 1\},$$

which is all of U when $S_2 = 0$ and otherwise an open halfspace.

A projective transformation $T\colon Y \to V$ is of the form

(2) $$T(\cdot) = \frac{T_o(\cdot) + T_1}{1 - \langle T_2, \cdot \rangle}$$

where $T_o\colon Y \to V$ is linear, $T_1 \in V$ and $T_2 \in X$. The transformations S and T given by (1) and (2) respectively are said to be mutually <u>adjoint</u> if S_o and T_o are adjoint, $S_1 = T_2$ and $S_2 = T_1$. To

see that this definition is meaningful, note that S_0, S_1, and S_2 are uniquely determined by S. That is obvious in the case of $S_1 = S(0)$. For S_0 and S_2 it follows from a routine calculation that can be simplified by reduction to the case in which $S_1 = 0$.

Note that a projective transformation is linear if and only if its adjoint is linear, and is affine if and only if its adjoint preserves the origin.

THEOREM 1. *The projective transformations* $S:U \to X$ *and* $T:Y \to V$ *are mutually adjoint if and only if the following two conditions are satisfied.*

(a) $\text{dmn } S = \{u \in U: \langle u, T(0) \rangle < 1\}$ *and* $\text{dmn } T = \{y \in Y: \langle S(0), y \rangle < 1\}$;

(b) *for all* $u \in \text{dmn } S$ *and* $y \in \text{dmn } T$,

$$\langle S(u), y \rangle \leq 1 \iff \langle u, T(y) \rangle \leq 1.$$

Proof. Plainly (a) holds if and only if $S_1 = T_2$ and $S_2 = T_1$. Suppose, then, that $S_1 = T_2 = x \in X$ and $S_2 = T_1 = v \in V$, and note that for each $u \in \text{dmn } S$ and $y \in \text{dmn } T$,

$$\langle S(u), y \rangle \leq 1 \iff \langle S_0(u), y \rangle + \langle u, v \rangle + \langle x, y \rangle \leq 1$$

and

$$\langle u, T(y) \rangle \leq 1 \iff \langle u, T_0(y) \rangle + \langle u, v \rangle + \langle x, y \rangle \leq 1.$$

If S and T are adjoint then so are S_0 and T_0, whence $\langle S_0(u), y \rangle = \langle u, T_0(y) \rangle$ and condition (b) is satisfied. Suppose, on the other hand, that S and T are not adjoint, whence there exist $u \in U$ and $y \in Y$ such that

(3) $$\langle S_o(u),y \rangle \neq \langle u,T_o(y) \rangle.$$

In attempting to show that (b) fails we may assume $x \neq 0$, for a similar argument applies when $v \neq 0$ and the case in which v and x are both 0 is obvious. With $x \neq 0$, a simple one-dimensional argument shows y may be replaced by a "nearby" point of Y so as to preserve (3) and have $\langle x,y \rangle \neq 0$. It is then possible, replacing y by $y/\langle x,y \rangle$ and u by a sufficiently small positive multiple of u or of $-u$, to preserve (3) while obtaining

$$\max\{\langle S_o(u),y \rangle, \langle u,T_o(y) \rangle\} + \langle u,v \rangle > 0$$

and

$$\langle u,v \rangle < 1 = \langle x,y \rangle.$$

We may assume

(4) $$\langle S_o(u),y \rangle < \langle u,T_o(y) \rangle,$$

for a similar argument applies under the reverse inequality. Since

(5) $$\langle u,T_o(y) \rangle + \langle u,y \rangle + \langle x,y \rangle > 1,$$

y may be replaced by $(1-\epsilon)y$, for a sufficiently small $\epsilon > 0$, so as to preserve (4) and (5) and obtain $\langle x,y \rangle < 1$. But then

(6) $$u \in \mathrm{dmn}\ S \quad \text{and} \quad y \in \mathrm{dmn}\ T,$$

and in view of (4) and (5), u may be replaced by ηu for some $\eta \in\ ]0,1[$ so as to preserve (6) and obtain

$$\langle S_o(u),y \rangle + \langle u,v \rangle + \langle x,y \rangle < 1 < \langle u,T_o(y) \rangle + \langle u,v \rangle + \langle x,y \rangle.$$

That completes the proof of Theorem 1.

THEOREM 2. <u>For</u> <u>each</u> <u>pair</u> <u>of</u> <u>transformations</u> $S:U \to X$ <u>and</u> $T:Y \to V$ <u>satisfying</u> <u>condition</u> (b), <u>and</u> <u>for</u> <u>each</u> <u>set</u> $W \subset$ dmn S, <u>it</u> <u>is</u> <u>true</u> <u>that</u>

$$T((SW)^O \cap (dmn\ T)) = W^O \cap (rng\ T).$$

Proof. Observe that for all $y \in$ dmn T and $v = T(y) \in$ rng T, the following five statements are equivalent:

$$v \in W^O; \quad w \in W \Rightarrow \langle w, v \rangle \le 1; \quad w \in W \Rightarrow \langle w, T(y) \rangle \le 1;$$

$$w \in W \Rightarrow \langle S(w), y \rangle \le 1; \quad y \in (SW)^O.$$

In the applications of the next section, Theorem 2's statement of equality is called the <u>basic</u> <u>formula</u>.

Face-figures of convex polytopes. The remaining discussion is
restricted to a finite-dimensional Euclidean space E with the
usual inner product. The notion of F-figure defined in the
Introduction extends the well-known notion of a vertex-figure
[2, p.49].

Now suppose that P and F are given, and H and H' are
two hyperplanes determining F-figures of P. For each (d+1)-face
G of P, the intersections $G \cap H$ and $G \cap H'$ are both empty or are
d-faces of $P \cap H$ and $P \cap H'$ respectively, and since all d-faces of
$P \cap H$ and $P \cap H'$ arise in this way it follows that the two F-figures
are combinatorially equivalent polytopes. When F consists of
a single vertex, the vertex-figures $P \cap H$ and $P \cap H'$ are sections
of the same cone and hence are projectively equivalent [2, p.49].
For higher-dimensional F the projective relationship among
F-figures is more complicated, and is described in Corollary 4 below.

A polar pair of convex bodies in E is a pair (C,D) of
compact convex sets such that $C^o = D$ and $D^o = C$; this implies the
origin is interior to both C and D. When L is a linear subspace
of E and C is a convex body in E with $0 \in$ int C, let C^L
denote the polar of $C \cap L$ relative to L; that is $C^L = C^o \cap L$.

The following result is easy to prove directly [4, p.91][5, p.70],
but we derive it from the basic formula to illustrate the use of the
formula.

COROLLARY 1. <u>If</u> (C,D) <u>is a polar pair of convex bodies in</u> E, L <u>is a linear subspace of</u> E, <u>and</u> S <u>is the orthogonal projection of</u> E <u>onto</u> L, <u>then</u>

$$SD = C^L.$$

Proof. Let $T:L \rightarrow E$ be the adjoint of the linear transformation $S:E \rightarrow L$, and let $W = D$. Since T is merely the injection of L into E, it follows from the basic formula that

$$(SD)^L = C \cap L.$$

The desired conclusion is obtained by taking polars of both sides relative to L.

Now suppose that C and D are polytopes, and let $\ell = \dim L$. When $\ell < \dim E$ it follows from [1, p.168] that C^L is the image under S of the ℓ-skeleton of D, and from [6, p.471] that there is an ℓ-dimensional subcomplex of D's boundary complex that projects biuniquely onto C^L. The next result describes a situation in which it is easy to find such a subcomplex.

COROLLARY 2. <u>Suppose that</u> (P,Q) <u>is a polar pair of polytopes in</u> E, F <u>is a proper face of</u> P, <u>and</u> H <u>is a hyperplane strictly separating</u> F <u>from all vertices of</u> P~F. <u>Let</u> $\underset{\sim}{C}(Q,F)$ <u>denote the cell-complex formed by all facets of</u> Q <u>that contain</u>

$$F^Y = \{q \in Q: f \in F \Rightarrow \langle f,q \rangle = 1\},$$

<u>the face of</u> Q <u>conjugate to</u> F <u>with respect to the polarity.</u> <u>If</u> $O \in H$ <u>then the orthogonal projection of</u> E <u>onto</u> H <u>carries</u> $\cup \underset{\sim}{C}(Q,F)$ <u>biuniquely onto</u> P^H.

Proof. Let J^+ denote the open halfspace in E that contains F and is bounded by H, and let J^- denote the opposite open halfspace $-J^+$. Let $R^{\pm}$ denote the open ray that issues from O, is orthogonal to H, and lies in $J^{\mp}$, so that the closure of $R^{\pm}$ is the polar of $J^{\pm}$. The polytope Q is the intersection of the closed halfspaces polar to the segments $[O,v]$, where v is a vertex of P, and for each such v the conjugate facet v^{γ} of Q lies in the bounding hyperplane of the halfspace. Note that

$$v \in F \;\Rightarrow\; v \in J^+ \;\Rightarrow\; [0,v]^O \supset R^+$$

$$v \notin F \;\Rightarrow\; v \in J^- \;\Rightarrow\; [0,v]^O \supset R^-$$

Since $\underset{\sim}{C}(Q,F)$ is formed from all facets v^{γ} of Q such that v is a vertex of F, it follows that $\cup\underset{\sim}{C}(Q,F)$ is the "lower boundary" of Q in the sense that

$$\cup\underset{\sim}{C}(Q,F) = \{q \in \partial Q: (q+R^-)\cap Q = \emptyset\}.$$

Hence the orthogonal projection of E onto H carries $\cup\underset{\sim}{C}(Q,F)$ biuniquely onto the image of Q, and by Corollary 1 that image is P^H.

When S is a translation of E such that $O \in \text{int } SP$, the polar $(SP)^O$ is of course combinatorially equivalent to the polar $P^O = Q$. Thus Corollary 2 implies that, for a given F and P, all F-figures of P have boundary complexes combinatorially equivalent to the boundary of the complex $\underset{\sim}{C}(Q,F)$. To establish a stronger, projective result, we use the following variant of [5, p.67], which extends a result of [2, p.48].

2.4

COROLLARY 3. Suppose that $S:E \to E$ is a projective transformation, S^a is its adjoint, and C is a compact convex set in E such that $C \subset$ dmn S and $O \in$ (int C)$\cap$(int SC). Then

$$S^a((SC)^o) = C^o.$$

Proof. By the basic formula,

$$S^a((SC)^o \cap (dmn\ S^a)) = C^o \cap (rng\ S^a).$$

But $O \in$ int C and S is certainly nonsingular, so $S_1 = S(O) \in$ int SC. Since dmn S^a = {$y \in E: \langle S_1, y \rangle < 1$}, it follows that $(SC)^o \subset$ dmn S^a and hence

$$S^a((SC)^o)\ = C^o \cap (rng\ S^a).$$

But $(SC)^o$ is compact and convex, whence the same is true of $S^a((SC^o))$. Since C^o is compact and convex while rng S^a is open, it then follows readily that $C^o \subset$ rng S^a and the proof is complete.

We are now able to explain the projective relationship among the various face figures associated with a given face of a polytope. For a relative interior point k of a compact convex set K in E, let K^k denote the polar relative to the flat aff K with origin translated to k; that is,

$$K^k\ =\ k + \{m \in (aff\ K) - k: \ell \in K - k \implies \langle \ell, m \rangle \leq 1\}.$$

By Corollary 3 as applied to translations (or by [2, p.48]) the relative polars K^k are all projectively equivalent as k ranges over relint K. When n = 1 the following becomes the known result on vertex-figures.

COROLLARY 4. Let the notation be as in Corollary 2, except
that $0 \in H$ is not required and hence the intersection $K = P \cap H$
may be an arbitrary F-figure of P. Then for each $k \in$ relint K
the relative polar K^k is the biunique image of $U \underline{C}(Q,F)$ under a
projective transformation. Thus when P and F are given and n
is the number of vertices of F, any two relative polars of
F-figures of P are projectively n-equivalent in the sense that
each can be partitioned into n polytopes so that for $1 \leq i \leq n$ the
i^{th} polytope in the first partition is projectively equivalent to
the ith polytope in the second partition.

Proof. For each $u \in E$, let $S(u) = u - k$. By Corollary 3,
$S^a((SP)^o) = Q$ and hence Q is carried onto $(SP)^o$ by the pro-
jective transformation M inverse to S^a. By Corollary 2, the
set $M(U \underline{C}(Q,F))$ maps biuniquely onto K^k under the orthogonal
projection Π of E onto H, and ΠM is then a projective
transformation mapping $U \underline{C}(Q,F)$ biuniquely onto K^k. For the
final statement of Corollary 4 note that for each facet G of
Q in $\underline{C}(Q,F)$, the restriction of ΠM to G admits a projective
inverse.

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